MATHEMAGIC ACTIVITY BOOK

CLASS - XI

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SYLLABUS GUIDELINES
CLASS - XI

Based on CBSE & ICSE Syllabus

UNIT-I: SETS AND FUNCTIONS

1. **Sets:**
   Sets and their representations. Empty set. Finite & Infinite sets. Equal sets. Subsets. Subsets of the set of real numbers especially intervals (with notations). Power set. Universal set. Venn diagrams. Union and Intersection of sets. Difference of sets. Complement of a set.

2. **Relations & Functions:**
   Ordered pairs, Cartesian product of sets. Number of elements in the cartesian product of two finite sets. Cartesian product of the reals with itself (upto R R R ´´).
   Definition of relation, pictorial diagrams, domain, co-domain and range of a relation.
   Function as a special kind of relation from one set to another. Pictorial representation of a function, domain, co-domain & range of a function. Real valued function of the real variable, domain and range of these functions, constant, identity, polynomial, rational, modulus, signum and greatest integer functions with their graphs. Sum, difference, product and quotients of functions.

3. **Trigonometric Functions:**
   Positive and negative angles. Measuring angles in radians & in degrees and conversion from one measure to another. Definition of trigonometric functions with the help of unit circle. Truth of the identity $\sin^2 x + \cos^2 x = 1$, for all x. Signs of trigonometric functions and sketch of their graphs. Expressing sin (x+y) and cos (x+y) in terms of sinx, siny, cosx & cosy. Deducing the identities like following :

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \cot(x \pm Y) = \frac{\cot x \cot y \pm 1}{\cot x \mp \cot y},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}, \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

   Identities related to sin 2x, cos2x, tan 2x, sin3x, cos3x and tan3x. General solution of trigonometric equations of the type sinq = sina, cosq = cosa and tanq = tan a. Proofs and simple applications of sine and cosine formulae.

UNIT – II: ALGEBRA

1. **Principle of Mathematical Induction:**
   Processes of the proof by induction, motivating the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

2. **Complex Numbers and Quadratic Equations**:
   Need for complex numbers, especially $\sqrt{-1}$, to be motivated by inability to solve every quadratic equation. Brief description of algebraic properties of complex numbers.
   Argand plane and polar representation of complex numbers. Statement of Fundamental Theorem of Algebra, solution of quadratic equations in the complex number system.

3. Linear Inequalities:
Linear inequalities. Algebraic solutions of linear inequalities in one variable and their representation on the number line. Graphical solution of linear inequalities in two variables. Solution of system of linear inequalities in two variables- graphically.

4. Permutations & Combinations:
Fundamental principle of counting. Factorial n. Permutations and combinations, derivation of formulae and their connections, simple applications.

5. Binomial Theorem:
History, statement and proof of the binomial theorem for positive integral indices. Pascal's triangle, general and middle term in binomial expansion, simple applications.

6. Sequence and Series:
Sequence and Series. Arithmetic progression (A. P.), arithmetic mean (A.M.). Geometric progression (G.P.), general term of a G. P., sum of n terms of a G.P., geometric mean (G.M.), relation between A.M. and G.M. Sum to n terms of the special series: $\sum n, \sum n^2$ and $\sum n^3$.

UNIT- III: COORDINATE GEOMETRY

1. Straight Lines:
Brief recall of 2D from earlier classes. Slope of a line and angle between two lines. Various forms of equations of a line: parallel to axes, point-slope form, slope-intercept form, two -point form, intercepts form and normal form. General equation of a line. Distance of a point from a line.

2. Conic Sections:
Sections of a cone: circles, ellipse, parabola, hyperbola, a point, a straight line and pair of intersecting lines as a degenerated case of a conic section. Standard equations and simple properties of parabola, ellipse and hyperbola. Standard equation of a circle.

3. Introduction to Three -dimensional Geometry
Coordinate axes and coordinate planes in three dimensions. Coordinates of a point in space. Distance between two points and section formula.

UNIT-IV: CALCULUS

1. Limits and Derivatives:
Derivative introduced as rate of change both as that of distance function and geometrically, intuitive idea of derivatives. Definition of derivative, limits, limits of trigonometric functions.

UNIT-V: MATHEMATICAL REASONING

1. Mathematical Reasoning:
Mathematically acceptable statements. Connecting words/ phrases - consolidating the understanding of "if and only if (necessary and sufficient) condition", "implies", "and/or", "implied by", "and", "or", "there exists" and their use through variety of examples related to real life and Mathematics. Validating the statements involving the connecting words-difference between contradiction, converse and contrapositive.

UNIT-VI: STATISTICS & PROBABILITY

1. Statistics:
Measure of dispersion; mean deviation, variance and standard deviation of ungrouped/grouped data. Analysis of frequency distributions with equal means but different variances.

2. Probability:
Random experiments: outcomes, sample spaces (set representation). Events: occurrence of events, 'not', 'and' & 'or' events, exhaustive events, mutually exclusive events. Axiomatic (set theoretic) probability, connections with the theories of earlier classes. Probability of an event, probability of 'not', 'and' & 'or' events.

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Mathematics for Everyone

• What is a number system?
The number system consists of some important sets of numbers.

• Natural numbers : The numbers 1, 2, 3, ... are called the natural or counting numbers. The set of natural numbers is denoted by *N*. *i.e.* $N = \{1, 2, 3, ...\}$.

• Integers : The numbers 0, ±1, ±2, ... are called integers. The set of integers is denoted by *Z*. *i.e.* $Z = \{.... -3, -2, -1, 0, 1, 2, 3, ...\}$.
The set of positive integers are known as Z^+ and the set of negative integers are known as Z^-
The set of integers divisible by 2 is called *even integers* and the set of integers which are not divisible by 2 are called *odd integers*. 0 (zero) is an *even integer*.

• Prime and composite integer : A positive integer which has no divisors other than 1 and itself is called a *prime number* and others are called *composite integers*. 1 is neither prime nor composite. There are infinitely many prime numbers.

• Relatively prime number : The number *p* and *q* are called relatively prime number if h.c.f. of *p* and *q* is 1. *e.g.* 2 and 9 are relatively prime.

• Rational and irrational number : A number *r* is rational if it can be written as a fraction $r = p/q$, where *p* and *q* both are integers. In reality each number can be written in many different ways. But to be rational, a number ought to have at least one fractional representation.

e.g. the number $\left(\frac{\sqrt{5}+1}{2}\right)^2 + \left\{\frac{(\sqrt{5}-1)}{2}\right\}^2$ may not look like a rational but it simplifies to 3 = 3/1 and hence a rational number.
The set of rational numbers is denoted by *Q* where $Q = \{p/q,\ p,\ q$ are integers, $q \neq 0\}$.
The set *Q* of all rational numbers is equivalent to set *N* of all integers, where as the number $\sqrt{5}$ is called an irrational number. However the theory of irrational number belongs to calculus. Although with the help of arithmetic we may prove that $\sqrt{5}$ is not rational.

So, the numbers like $\sqrt{2}$, $\sqrt{5}$, $3^{1/5}$, π, *e*, log6 etc are irrational numbers.

Real and complex numbers : Collectively rational and irrational numbers are called *real*. Thus a set obtained by taking all rational and irrational numbers is called a *set of real numbers* and it is denoted by *R*. While it is true that there are infinitely many rational numbers and infinitely many irrational numbers, in a well defined sense, there are more irrational numbers than rational.
Real numbers can also be treated as points on a line (also called the real line). Now between any numbers of one kind there always existed numbers of the same and of the other kind. However rational numbers form a countable set

where as irrational numbers form a set which is not countable.
[A set A is called countable (or enumerable, or denumerable and even sometimes may be numerable) if $A \sim N$, *i.e.* if it is equivalent to set of all integers).

Complex numbers are pairs $C = (x, y)$ of two real numbers. Since the situations of having to handle two numbers instead of one is quite complex, the terminology is well justified. The impetus for developing the theory of complex numbers came originally from unsolvability of a simple equation $x^2 + 1 = 0$ in the set of real numbers. In the set of complex numbers the equation has two solutions; $\pm i$. The set of the complex number could also be defined by adding this new number.
Thus the number of the form $a + ib$ where $i = \sqrt{-1}$ and a, b are real numbers are known as *complex number*. Thus without breaking existing laws and by adding a single new number we actually get a set of numbers in which every polynomial equation with real coefficients has a solution. Complex numbers have unusual properties especially when it comes to differentiation. With real numbers differentiation is a starting point of calculus and real analysis. With complex numbers it leads to the analytic function theory.

Imaginary numbers : As we have defined a complex number C is a pair of (x, y) of two real numbers, x in the pair called the *real part* and y in the pair is the *imaginary part*.
Now $C = x + iy$ be a complex number;
here $x \rightarrow$ real part ; $y \rightarrow$ imaginary part.

Algebraic and transcendental numbers.
Let us consider polynomial with whole (+ve integers and −ve integers) coefficients.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots ax + a_0 = 0.$$

Real roots of such equations are said to be algebraic. In other words, a number a is called algebraic if it satisfies an algebraic equation $P_n(a) = 0$, for some polynomial $P_n(x) = 0$ with integer coefficients. Rational numbers and integers are all algebraic. Indeed, for given rational number $r = p/q$, if we take $n = 1$ and $P_1(x) = qx - p$. Obviously $P_1(r) = 0$. $\sqrt{5}$ is also algebraic for it solves the equation $P_2(x) = x^2 - 5 = 0$. It can be proved that, for every n, the set of polynomials of order n is countable. *The fundamental theorem of algebra claims that a polynomial of order n has exactly n (perhaps complex) roots*. Then the set of all numbers that satisfy some equation of order n is countable. The subset of this set that consists of real number is as well countable. Thus the set of all algebraic numbers is countable.

Transcendantal numbers : Real numbers that are not algebraic are called *transcendantal*. Thus the set R of all real numbers is the union of two sets - algebraic and transcendantal. The set R is not countable and therefore the set of all transcendantal numbers is not countable either. π and e are transcendantal numbers.

Activity work sheet

Perfect numbers

A perfect number is a number like 6 with the property that all its factors add up to the number itself.

e.g. the factors of 6 are 1, 2 and 3 and $1 + 2 + 3 = 6$.

The next smallest perfect number is 28. The factors of 28 are 1, 2, 4, 7 and 14 and $1 + 2 + 4 + 7 + 14 = 28$.

i.e. any number n is perfect if it equals the sum of its divisors including 1 but excluding n.

The next four perfect numbers are 496, 8128, 33550336, 8589869056.

Only 37 perfect numbers have been found. All even perfect numbers must have the form $2^{p-1}(2^p - 1)$ where p and 2^{p-1} are prime numbers. *e.g.* for 6 and 28 we have $p = 2$ and $p = 3$. No one has yet discovered an odd perfect number but it is known that there is no odd perfect number less than 10^{300}.

1. What is special about this number 73939133.

Ans.: All the numbers 73939133
7393913
739391
73939
7393
739
73
7

are prime numbers. This is the biggest number with such property.

2. What are *twin primes*?

Ans.: The number of the form $n - 1$ and $n + 1$, both of which are primes are called *twin primes*.

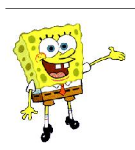
e.g. 17 and 19 or 101 and 103.

The world's largest known pair of twin primes is

$190116 \times 3003 \times 10^{5120} - 1$ and

$190116 \times 3003 \times 10^{5120} + 1$.

This was discovered in 1995. It is believed that there are infinitely many prime numbers.



Lets do Some Number Magic

Mathematics is not always high and dry. You can play and have fun with math too. You may ask any person to think any 3 digited number and may guess its number. How?? Here it follows.

How to guess the numbers of 3 digits.

Let the number be ABC, where A, B and C are the three digits. Find out the remainders from division by 11 of the numbers formed by the digits ABC, BCA, CAB in turn. Suppose the respective remainders be a, b & c. Do the sum $a + b$, $b + c$ & $c + a$. If any of these is odd, increase or decrease it by 11 in order to get a positive number less than 20. Divide each of these number by 2 and the obtained numbers are the digits A, B, C in order.

Example : Let the number thought by your friend is 789. You are supposed to guess its number.

$A = 7$, $B = 8$, $C = 9$

$ABC = 789$, $BCA = 897$, $CAB = 978$

$$11\,\begin{array}{|c|}789\\77\end{array}\,71 \qquad 11\,\begin{array}{|c|}897\\88\end{array}\,81 \qquad 11\,\begin{array}{|c|}978\\88\end{array}\,88$$

$$\begin{array}{c}19\\ \underline{11}\\ 8\end{array} \qquad \begin{array}{c}17\\ \underline{11}\\ 6\end{array} \qquad \begin{array}{c}98\\ \underline{88}\\ 10\end{array}$$

$$a = 8 \qquad b = 6 \qquad c = 10$$

$a + b = 14$, $b + c = 16$, $c + a = 18$

Now your friend after performing the above operations tells you

$A = 14/2 = 7$, $B = 16/2 = 8$, $C = 18/2 = 9$.

So by the given procedure we obtain the 3 numbers 7, 8 & 9 in turn.

NUMBER FUN

1. Find a number such that each digit except zero appears just once in the number and its square. There are two solutions

(a) $(567)^2 = 321{,}489$

(b) $(854)^2 = 729{,}316$

2. Funny number sequence:

$$1 + 4 + 5 + 5 + 6 + 9 = 2 + 3 + 3 + 7 + 7 + 8$$

$$1^2 + 4^2 + 5^2 + 5^2 + 6^2 + 9^2 = 2^2 + 3^2 + 3^2 + 7^2 + 7^2 + 8^2$$

$$1^3 + 4^3 + 5^3 + 5^3 + 6^3 + 9^3 = 2^3 + 3^3 + 3^3 + 7^3 + 7^3 + 8^3$$

3. Funny square number: $(69)^2 = 4761$

It is interesting to see that the product 4761 has same digits of Kaprekar's well known constant number 6174.

NUMBER PUZZLE

Find four numbers such that the product of any two of them is one less than a perfect square. This problem belongs to famous mathematician *Diophantus*.

Solution : Let a, b, c and d are the required numbers. There are six possible products of any two of these, namely ab, ac, ad, bc, bd, cd. Then if m and n are any arbitrary integers, the requirement can be stated.

$a = m$; $b = n(mn + 2)$; $c = (n + 1)(mn + m + 2)$

$d = 4(mn + 1)(mn + m + 1)(mn^2 + mn + 2n + 1)$

$ab + 1 = (mn + 1)^2$; $ac + 1 = (mn + n + 1)^2$

$ad + 1 = (2m^2n^2 + 2m^2n + 4mn + 2m + 1)^2$

$bc + 1 = (mn^2 + mn + 2n + 1)^2$

$bd + 1 = (2m^2n^3 + 2m^2n^2 + 6mn^2 + 4mn + 4n + 1)^2$

$cd + 1 = (2m^2n^3 + 4m^2n^2 + 6mn^2 + 2m^2n + 8mn + 4n + 2m + 3)^2$

m and n are any integers.

For example, $m = 2$ and $n = 1$ yield $a = 2$, $b = 4$, $c = 12$, $d = 420$, $ab + 1 = 3^2$, $ac + 1 = 5^2$, $ad + 1 = 29^2$, $bc + 1 = 7^2$, $bd + 1 = 41^2$, $cd + 1 = 71^2$.

PERFECT SQUARES

Since $a = \frac{10^n - 1}{9}$, $10^n - 1 = 9a$

$b = 9a + 1 = 10^n$. Let $c = 8a + 1$.

Now, $4ab + c = 4a(9a + 1) + 8a + 1$

$= 36a^2 + 12a + 1 = (6a + 1)^2$, $a = 1, 11, 111, \ldots$

$= (7^2, 67^2, \ldots..)$, Put $a = 1$, $6a + 1 = 7$,

Put $a = 11$, $6a + 1 = 6 \times 11 + 1 = 67$

$(a - 1)b + c = (a - 1)(9a + 1) + 8a + 1$

$= 9a^2 - 9a + a - 1 + 8a + 1 = 9a^2 = (3a)^2$

$16ab + c = 16a(9a + 1) + (8a + 1)$

$= 144a^2 + 24a + 1 = (12a + 1)^2$

$(25a + 3)b + c = (25a + 3)(9a + 1) + 8a + 1$

$= 225a^2 + 52a + 3 + 8a + 1 = 225a^2 + 60a + 4$

$= (15a + 2)^2$.

Amazing Maths

How to find last digit of a number with high power & the remainder left when such number is divided by some small number.

Since last few years, in many engineering and other entrance objective exams (like UPSEAT & MBA/MCA etc.), questions relating to "find the last/unit digit of a high powered number" is being asked. It has been observed that most of the students do these problems using the concept of Binomial Theorem, which becomes lengthy and time consuming in objective exams.

So, a much easier and time saving method of doing these problems is presented here.

First you should understand that Mod (or Modulo) is a mathematical function that gives remainder of the process of division and the statement that "x is the remainder when N is divided by b" is symbolically denoted as $N \equiv x$ (Mod b). Now, to find the last digit of a number (N), we can simply find the remainder (x) obtained when N is divided by 10 (b in this case).

Now, one can practise the method by going through few solved examples:

Examples :

- Find the last digit of 3^{123}.

$3^4 \equiv 1$ (Mod 10) $\Rightarrow (3^4)^{30} \equiv 1^{30}$ (Mod 10)

or, $3^{120} \equiv 1$(Mod 10) .... (i)

Again, $3^3 \equiv 7$ (Mod 10)... (ii)

Eqn. (i) × (ii) gives, $3^{120} \times 3^3 \equiv 1 \times 7$ (Mod 10)

or $3^{123} \equiv 7$ (Mod 10)

$\therefore$ Last digit = 7.

[* ***Note*** : When number (N) is raised to any power, the remainder (x) is also raised to the same power].

- Last digit of 11^{132} = ? ; $11 \equiv 1$ (Mod 10)

$\therefore$ $11^{132} \equiv 1^{132}$ (Mod 10) or,

$11^{132} \equiv 1$ (Mod 10)

$\therefore$ last digit = 1.

- Last digit of 6^{500} = ?

$6^2 \equiv 6$ (Mod 10)

$\therefore$ $6^4 \equiv 6^2$ (Mod 10) $\equiv 6$ (Mod 10).

[***Note*** : Since 6^2 gives 6 as remainder on division by 10]

Similary, $(6^2)^{250} \equiv 6$ (Mod 10).

or, $6^{500} \equiv 6$ (Mod 10)

$\therefore$ Last digit = 6.

- Last digit of 7^{291} = ?

$7^4 \equiv 1$ (Mod 10)

$\therefore$ $(7^4)^{72} \equiv 1^{72}$ (Mod 10)

or, $7^{288} \equiv 1$ (Mod 10) .... (i)

Also, $7^2 \equiv 9$ (Mod 10) .... (ii)

$7^1 \equiv 7$ (Mod 10) .... (iii)

Eqns. (i) × (ii) × (iii) gives,

$7^{291} \equiv 1 \times 7 \times 9$ (Mod 10)

$\equiv 63$ (Mod 10) $\equiv 3$ (Mod 10)

$\therefore$ Last digit = 3.

[*Note* : In general $N \equiv N$ (Mod b) if $N < b$ like in above example, 7 being less than 10, cannot be divided by 10. So, we say "7 on division by 10 leaves 7 (itself) as remainder" which is represented as "$7 \equiv 7$ (Mod 10)"].

Now the above idea can be extended to find the remainder when number with high power is divided by a smaller number.

Examples :

- When 7^{30} is divided by 5, what is the remainder?

$7^4 \equiv 1$ (Mod 5).

So, $(7^4)^7 \equiv 1^7$ (Mod 5)

or, $7^{28} \equiv 1$ (Mod 5) ... (i)

Also, $7^2 \equiv 4$ (Mod 5) ... (ii)

Eqn. (i) × (ii) gives, $7^{28} \times 7^2 \equiv 1 \times 4$ (Mod 5)

or, 7^{30} 4 (Mod 5)

$\therefore$ Remainder = 4.

Note : The above problem can be also solved by "Binomial Theorem" method as follows:

$7^{30} \equiv (7^2)^{15} = (49)^{15} = (50-1)^{15}$

$= 50^{15} - {}^{15}C_1 50^{14} + {}^{15}C_2 50^{13} - {}^{15}C_3 50^{12} + \ldots + ({}^{15}C_{15} (-1)^{15} 50^0)$

$= 50^{15} - {}^{15}C_1 50^{14} + {}^{15}C_2 50^{13} - {}^{15}C_3 50^{12} + \ldots + (-1)$

$= 50^{15} - {}^{15}C_1 50^{14} + {}^{15}C_2 50^{13} - {}^{15}C_3 50^{12} + \ldots + (-5+4)$

= (a multiple of 5) + 4.

$\therefore$ remainder = 4.

2. When 5^{20} is divided by 7, what is the remainder?

$5^2 \equiv 4$ (Mod 7)

$\therefore$ $(5^2)^2 \equiv 4^2$ (Mod 7)

or, $5^4 \equiv 16$ (Mod 7) $\equiv 2$ (Mod 7).

[*Note :* since 16, on division by 7 gives 2 as remainder]

$\therefore$ $(5^4)^5 \equiv 2^5$ (Mod 7)

or, $5^{20} \equiv 32$ (Mod 7) $\equiv 4$ (Mod 7)

$\therefore$ Remainder = 4.

[Similarly, 32 when divided by 7 gives 4 as remainder]

A more typical example

3. What is the remainder obtained when $64 \times 65 \times 66$ is divided by 67?

$64 \equiv -3$ (Mod 67) ... (i)

$65 \equiv -2$ (Mod 67) ... (ii)

$66 \equiv -1$ (Mod 67) ... (iii)

Eqns. (i) × (ii) × (iii) gives,

$64 \times 65 \times 66 \equiv -6$ (Mod 67) $\equiv \underset{(61=67-6)}{61}$ (Mod 67)*

$\therefore$ Remainder = 61.

[* *Note*: When (N) is divided by (b) such that $b > N$ then "$N \equiv -a$ (Mod b)" means "N is short of a to get divisible by b where $a = b - N$"

Here, a is not the remainder. While as told earlier the "remainder of division of N by b where $b > N$ will be N itself" *i.e.* "$N \equiv N$ (Mod b)" ($N = b - a$).